Table for Third-Degree Spline Interpolation Using Equi-Spaced Knots

By W. D. Hoskins

Abstract. A table is given for the calculation of the parameters of a third-degree natural spline with n data points ($n > 2$) using a minimum number of multiplications and divisions. In addition, an example is given that demonstrates the method of use and enables comparisons to be made with a method previously described.

1. Introduction. It has been previously demonstrated in [1] that for the special case of a natural spline of third degree, interpolating to equidistant data points, it is possible to determine the unknown parameters of the spline explicitly, without the need for solving a set of linear simultaneous equations. In the ensuing analysis, it is shown that by making use of an alternative, but equivalent, form for the spline, not only is the number of required tables halved, but the volume of computation needed to produce the spline is reduced. An additional advantage of this preferred form is that it involves only local values, and hence less calculation is demanded to evaluate interpolated values from the computed spline.

2. Definitions. The cubic spline $S(x)$, interpolating to the values (x_j, y_j) for $j = 1(1)n$, is defined to be a cubic polynomial in each interval $x_i \leqq x \leqq x_{i+1}$ ($i = 1(1)n - 1$) such that $S(x) \in C^2$. A cubic spline is further called natural, if

$$S''(x_1) = S''(x_n) = 0,$$

for in this case the interpolating function is the smoothest, in the sense that the integral $\int_{x_1}^{x_n} [S''(x)]^2\, dx$ is minimized. We choose to denote the second derivatives of $s(x_j)$ on the uniform mesh h by M_j, in which case the cubic spline $S(x)$ is given uniquely in the interval $[x_i, x_{i+1}]$ by

$$(2.1)\qquad \begin{aligned} S(x) = {} & \left(y_i - \frac{h^2}{6} M_i\right) \frac{(x_{i+1} - x)}{h} + \left(y_{i+1} - \frac{h^2}{6} M_{i+1}\right) \frac{(x - x_i)}{h} \\ & + \frac{h^2}{6} M_i \left(\frac{x_{i+1} - x}{h}\right)^3 + \frac{h^2}{6} M_{i+1} \left(\frac{x - x_i}{h}\right)^3. \end{aligned}$$

The requirement that $S(x) \in C^2$ enforces that the additional constraint

$$(2.2)\qquad \frac{h^2}{6} M_{j+1} + 4\frac{h^2}{6} M_j + \frac{h^2}{6} M_{j-1} = \delta^2 y_j, \qquad (j = 2(1)n - 1),$$

called the continuity equation, is satisfied ([2] and [3]).

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3. Derivation of Formulae. From the definition of a natural spline, the boundary conditions

$$M_1 = M_n = 0 \tag{3.1}$$

are applicable and yield, when taken in conjunction with the continuity Eq. (2.2), the matrix equation

$$\begin{bmatrix} 4 & 1 & 0 & \cdots & 0 \\ 1 & 4 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & 4 & 1 \\ 0 & \cdots & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} \frac{h^2}{6} M_2 \\ \frac{h^2}{6} M_3 \\ \vdots \\ \frac{h^2}{6} M_{n-2} \\ \frac{h^2}{6} M_{n-1} \end{bmatrix} = \begin{bmatrix} \delta^2 y_2 \\ \delta^2 y_3 \\ \vdots \\ \delta^2 y_{n-2} \\ \delta^2 y_{n-1} \end{bmatrix} \tag{3.2}$$

for the unknowns $(h^2/6)M_j$ $(j = 2(1)n - 1)$.

The Eqs. (3.2) are tridiagonal and symmetric, thus, the solution is particularly simple using a single LU decomposition. Performing this decomposition gives the modified matrix equation,

$$\begin{bmatrix} \alpha_{n-3} & 0 & \cdots & & 0 \\ 1 & \alpha_{n-4} & \ddots & & \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \alpha_1 & 0 \\ 0 & \cdots & 0 & 1 & \alpha_0 \end{bmatrix} \frac{h^2}{6} \mathbf{M} = \begin{bmatrix} d_2 \\ d_3 \\ \vdots \\ d_{n-2} \\ d_{n-1} \end{bmatrix}, \tag{3.3}$$

where

$$\frac{h^2}{6} \mathbf{M} = \left\{ \frac{h^2}{6} M_2, \frac{h^2}{6} M_3, \cdots, \frac{h^2}{6} M_{n-2}, \frac{h^2}{6} M_{n-1} \right\}^T$$

with

$$\begin{aligned} \alpha_j &= 4 - 1/\alpha_{j-1}, \qquad (j = 1, 2, \cdots), \\ \alpha_0 &= 4. \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} d_{j-1} &= \delta^2 y_{j-1} - d_j/\alpha_{n-j-1} \qquad (j = n - 1(-1)3), \\ d_{n-1} &= \delta^2 y_{n-1}. \end{aligned} \tag{3.5}$$

Now, we define

$$\alpha_j = a_j/b_j \tag{3.6}$$

and, by substitution in (3.4), obtain the two recurrence relations

(3.7) $$a_{j+2} = 4a_{j+1} - a_j \qquad (j \geqq -1,\ a_0 = 4,\ a_{-1} = 1)$$

and

(3.8) $$b_{j+1} = a_j,$$

which enable Eq. (3.6) to be rewritten, using formula (3.8), in the form

(3.9) $$\alpha_j = a_j/a_{j-1} \qquad (j \geqq 0).$$

The recurrence relation (3.5) may be written, when $j = 3$, as

(3.10) $$d_2 = \delta^2 y_2 + \sum_{r=3}^{n-1} \left\{ \left[\prod_4^{r+1} \left(-\frac{1}{\alpha_{n-p}} \right) \right] \delta^2 y_r \right\},$$

or with substitution from Eqs. (3.9),

(3.11) $$d_2 = \delta^2 y_2 + \sum_{r=3}^{n-1} (-1)^r \left(\frac{a_{n-r-2}}{a_{n-4}} \right) \delta^2 y_r.$$

Equation (3.11) can be further abbreviated to

(3.12) $$a_{n-4} d_2 = \sum_{r=2}^{n-1} (-1)^r a_{n-r-2} \delta^2 y_r$$

and then, by substitution into the first of Eqs. (3.3), used to give

(3.13) $$a_{n-3} \left\{ \frac{h^2}{6} M_2 \right\} = \sum_{r=2}^{n-1} (-1)^r a_{n-r-2} \delta^2 y_r.$$

TABLE 1

j	a_j
−1	1
0	4
1	15
2	56
3	209
4	780
5	2911
6	10864
7	40545
8	1 51316
9	5 64719
10	21 07560
11	78 65521
12	293 54524
13	1095 52575
14	4088 55776
15	15258 70529
16	56946 26340
17	2 12526 34831

From the Eqs. (3.13) and (3.1), it is apparent, therefore, that the continuity Eq. (2.2) now enables the remainder of the unknown M_j to be rapidly calculated, using as starting values M_1, M_2. Further, since the terms $(h^2/6)M_j$ appear in the spline Eq. (2.1), it is possible to work directly with the unknowns $(h^2/6)M_j$, rather than M_j, and reduce both the rounding errors and volume of computation.

4. Example. The constants a_j are determined from the recurrence relation (3.7) and are given in Table 1.

The example considered is that treated in [1] and characterized by the data j, y_j, appearing in the following table:

TABLE 2
Numerical Example of a Natural Spline
$(h = 1)$

j	y_j	$\delta^2 y_j$	$\frac{40545}{6} M_j$	$\frac{1}{6} M_j$
1	244.0	—	0	0
2	221.0	10.0	73245	1.8065112
3	208.0	13.0	112470	2.7739548
4	208.0	3.5	3960	0.0976693
5	211.5	1.0	13597.5	0.3353681
6	216.0	−1.5	−17805	−.4391417
7	219.0	−1.0	−3195	−.0788013
8	221.0	−1.5	−9960	−.2456530
9	221.5	−2.0	−17782.5	−.4385868
10	220.0	—	0	0

The method of calculating the parameters is as follows: the quantity $(a_7/6)M_2$ is computed using Eq. (3.13) and Table 1 to supply the constants a_j (division by a_7 is delayed until the remaining quantities $(a_7/6)M_j$ have been determined so as to reduce the effect of rounding errors):

$$40545(\tfrac{1}{6}M_2) = [10864(10.0) - 2911(13.0) + \cdots + 4(-1.5) - 1(2.0)],$$

$$40545(\tfrac{1}{6}M_2) = 73245.$$

Finally, the unknowns $(a_7/6)M_3, \cdots, (a_7/6)M_{n-1}$ are calculated, using

$$a_7\left(\frac{1}{6}M_{j+1}\right) = a_7\delta^2 y_j - 4\left(\frac{a_7}{6}M_j\right) - \left(\frac{a_7}{6}\right)M_{j-1}, \qquad j = 2, \cdots, n-1,$$

and the quantities M_j tabulated in Table 2.

Cubic spline interpolation is then made, if required, using Eq. (2.1) and Table 2.

5. Conclusions. The method outlined above is a condensed way of solving the $(n-2) \times (n-2)$ simultaneous equations for the unknown parameters of the spline, but still possesses the advantage of requiring only $(2n - 4)$ multiplications/divisions as compared with the $O(n^2)$ required by the method proposed in [1].

Use of the three-term recurrence relation (2.2) and M_1, M_2 to evaluate the other M_i, though less sensitive than the method in [1] used to calculate the spline coefficients, is still liable to increase rounding errors, unless the precaution of using exact calculation is followed. However, the Eq. (2.1), used subsequently for interpolation, is not sensitive to rounding errors and consequently is preferred, both for accuracy and speed, over the form used in [1].

Department of Computer Science
University of Manitoba
Winnipeg, Manitoba, Canada

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